March 30, 2017

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Definition. $(X, \mathcal{J})$ is *locally compact* if $\forall x \in X$ $\exists$ compact nbhd K of x, i.e., $x \in \mathring{K} \subset K$.

This definition is somewhat *atypical* in topology. There is another more natural one, but was not adopted in historical development.

$(X, \mathcal{J})$ has *local compact bases* if $\forall x \in X$ $\exists$ local base $\mathcal{K}_x$ at x consisting of compact nbhds of x.

Proposition. If $(X, \mathcal{J})$ is Hausdorff and locally compact then it has local compact bases.

Key Idea of Proof. Let $x \in X$.

Then $\exists$ compact K, $x \in \mathring{K} \subset K$

Now, $(K, \mathcal{J}|_K)$ is compact Hausdorff

By previous lecture, it is regular and every nbhd U of x has $U_1 \in \mathcal{J}|_K$,

$$x \in U_1 \subset \overline{U_1} \subset U \cap \mathring{K} \subset K$$

$\overline{U_1}$ is a compact nbhd inside U.

One-point Compactification

Let $(X, \mathcal{T})$ be locally compact Hausdorff.
Then $\exists$ compact Hausdorff $(X^*, \mathcal{T}^*)$ such that
(i) $X^* \setminus X$ is a singleton
(ii) $\mathcal{T} = \mathcal{T}^*|_X$
(iii) If X is noncompact then $\overline{X} = X^*$
If X is compact then $X^* \setminus X$ is isolated.

Proof. Pick $\infty \notin X$ and let $X^* = X \cup \{\infty\}$.
Define $\mathcal{T}^* = \mathcal{T} \cup \{\{\infty\} \cup (X \setminus K) : K \subset X \text{ is compact}\}$
Note that since X is Hausdorff, that K is compact implies $X \setminus K \in \mathcal{T}$.
It then is easy to check that $\mathcal{T}^*$ is a topology.
To show that $(X^*, \mathcal{T}^*)$ is Hausdorff, we only need to consider $x \neq \infty$ and $\exists$ compact nbhd K, $x \in \mathring{K} \subset K$
In this case, $x \in \mathring{K}$, $\infty \in \{\infty\} \cup (X \setminus K)$
To show $(X^*, \mathcal{T}^*)$ is compact, let $\mathcal{G} \subset \mathcal{T}^*$ with $\bigcup \mathcal{G} \supset X^*$. May assume that

$$\mathcal{G} = \mathcal{g} \cup \{\{\infty\} \cup (X \setminus K)\} \quad \text{where } \mathcal{g} \subset \mathcal{T}$$

Then $\bigcup \mathcal{g} \supset K$ and it has a finite subcover for K. So, $\mathcal{G}$ has a finite subcover for X^*.

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In the case that X is non-compact, for any nbhd of ∞, it is of the form

$$\{\infty\} \cup (X \setminus K) \neq \{\infty\}$$

Thus, $(\{\infty\} \cup (X \setminus K)) \cap X = X \setminus K \neq \emptyset$

It shows that $\infty \in \overline{X}$ and $\overline{X} = X^*$.

In the case that X itself is compact then $\{\infty\} \cup (X \setminus X) \in \mathcal{T}^*$

Thus ∞ has a nbhd $\{\infty\}$ which does not meet X. So, ∞ is isolated.

Notions of compactness There are a number of notions of compactness. We only mention three here. The details will be given in a separate notes.

Heine-Borel. Every open cover has a finite subcover.

Bolzano-Weierstrass Every infinite subset has a cluster point.

Sequentially compact Every sequence has a convergent subsequence.